BACK PAPER EXAM ISI BANGALORE JULY 2017

B MATH ALGEBRA II 100 POINTS

Notes.
(a) Justify all your steps.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
(c) By default, F denotes a field.

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1. [22 points] Let A denote the matrix

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 2 & -3 & 0 \\ -1 & 1 & -1 \end{pmatrix}$$

(i) Find the characteristic polynomial, the eigenvalues and the eigenvectors of A.
(ii) Find an invertible 3×3 matrix X such that XAX^{-1} is a diagonal matrix and verify by expicitly multiplying out that XAX^{-1} is indeed diagonal.
(iii) Using (ii) or otherwise calculate A^{1000}.
(iv) Find the minimal polynomial of A.

2. [12 points] Let $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ be orthonormal subsets of $\mathbb{R}^n$. Prove that there exists an orthogonal matrix A such that $Av_i = w_i$ for $i = 1, 2, \ldots, r$.

3. [12 points] Let $T\colon V \to V$ be a linear operator on a finite dimensional vector space. If T satisfies $T \circ T = T$, then prove that $V = \ker(T) \oplus \mathrm{im}(T)$.

4. [20 points] Let F be a field and let $N = (n_{ij})$ be an $m \times m$ matrix over F which is *strictly upper triangular*, i.e., $n_{ij} = 0$ for $i \geq j$. Prove that $N^m = 0$. Conversely if A is an $m \times m$ matrix and if there exists an integer $k > 0$ such that $A^k = 0$ then prove that there exists an invertible $m \times m$ matrix X such that XAX^{-1} is strictly upper triangular.

5. [4 points] Let A be a square matrix over $\mathbb{C}$ such that $A^3 + 5A^2 + 2A - 3I = 0$. Prove that every eigenvalue λ of A satisfies $\lambda^3 + 5\lambda^2 + 2\lambda - 3 = 0$.

6. [30 points] In each of the following cases, give an example satisfying the given property. Give very brief justifications.

(i) A 2×2 matrix over $\mathbb{C}$ which is not normal.
(ii) A symmetric 2×2 matrix whose associated symmetric bilinear form is neither positive semi-definite nor negative semi-definite.
(iii) A 2×2 invertible matrix B over $\mathbb{R}$ with all positive entries such that B cannot be written as X^tX for any invertible matrix X.
(iv) A matrix in $SO_3(\mathbb{R})$ which is not the identity matrix and whose eigenvalues are all real.

(v) The matrix of a reflection about a line through the origin in $\mathbb{R}^2$.